Compliant motion control for robot manipulators

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A practical, non-linear controller design methodology is presented for robot manipulators guaranteeing that the robot end-point follows an input command vector 'closely' when the robot is not constrained by the environment, and that the contact force is a function of the same input command vector (used in the unconstrained environment) when the robot is constrained by the environment. The controller is capable of 'handling' both types of constrained and unconstrained manoeuvres, and is robust to bounded uncertainties in the robot dynamics. The controller does not need any hardware or software switch for the transition between unconstrained and constrained manoeuvring. Stability of the environment and the manipulator taken as a whole has been investigated, and a bound for stable manipulation has been derived. For stability, there must be some initial compliancy either in the robot or in the environment. A unified approach to modelling robot dynamics is expressed in terms of sensitivity functions, as opposed to the lagrangian approach, allowing the incorporation of the dynamic behaviour of all the robot manipulator elements.

Notation

- A closed-loop mapping from r to f in Fig. 5
- d vector of the external force on the robot end-point (all vectors are $n \times 1$)
- e input trajectory vector
- E environment dynamics
- f vector of the contact force $[f_1 \ f_2 \ \dots \ f_n]^T$
- f_{∞} limiting value of the contact force for infinitely rigid environment
- G robot dynamics with positioning controller
- H compensator transfer function matrix (operating on the contact force, f)
- I_n identity matrix
- r input-command vector
- *n* degrees of the freedom of the robot n < 6
- *S* robot manipulator sensitivity (1/stiffness)
- T positive scalar
- V forward loop mapping from e to f in Fig. 5
- x vector of the environment deflection
- y vector of the robot end-point position
- y_{∞} limiting value of the robot position for rigid environment
- x_0 vector of the environment position before contact
- θ vector of the joint angles of the robot
- $\varepsilon_{e}, \varepsilon_{d}, \mu, \gamma$ positive scalars
 - ω_0 frequency range of operation (bandwidth)
 - α_i, β_i, ν positive scalars
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1. Introduction

Most assembly operations and manufacturing tasks require mechanical interactions with the environment or with the object being manipulated, along with 'fast' motion in free and unconstrained space (Mason 1981, Paul and Shimano 1976, Salisbury 1980). We plan to develop a control system such that the robot will be capable of manoeuvring in both constrained and unconstrained environments without any hardware and software switches in the transition period. In meeting the above goal, the objective is to provide a stabilizing dynamic compensator for the robot manipulator such that the following design specifications are satisfied:

- (i) The robot end-point follows an input-command vector, r, when the robot manipulator is free to move.
- (ii) The contact force, f, a function of the input command vector, r, when the robot is in contact with the environment. (In this paper, 'force' implies both force and torque and 'position' implies both position and orientation.)

The first design specification allows for free manipulation when the robot is not constrained. If the robot encounters the environment, then according to the second design specification, the contact force will be a function of the input command vector, r. Consequently, the system will have bounded and controllable contact force. Note that r is an input command vector that is used for both unconstrained and constrained manoeuvres. The end-point of the robot will follow r when the robot is unconstrained, while the contact force, f, will be a function of r (preferably a linear function for some bounded frequency range of r) when the robot is constrained.

2. Dynamic model of the robot with positioning controllers

In this section, a general approach will be developed to describe the dynamic behaviour of a large class of industrial and research robot manipulators having positioning controllers. The fact that most industrial manipulators already have some kind of positioning controller is the motivation behind our approach. Also, a number of methodologies exist for the development of robust positioning controllers for direct and non-direct robot manipulators (Slotine 1985, Vidyasagar and Spong 1985). The unified approach for modelling robot dynamics discussed here is expressed in terms of sensitivity functions. It allows us to incorporate the dynamic behaviour of all the elements of a robot manipulator in addition to the rigid body dynamics.

The end-point position of a robot manipulator that has a positioning controller is a dynamic function of its input trajectory vector, e, and the external force, d. Let G and S be two functions that show the robot end-point position is a function of the input trajectory, e and the external force, d. (The assumption that linear superposition in (1) holds for the effects of d and e is useful in understanding the nature of the interaction between the robot and the environment. This interaction is in a feedback form and will be clarified with the help of Fig. 3. We will note in § 4 that the results of the non-linear analysis do not depend on this assumption, and one can extend the obtained results to cover the case when G[e] and S[d] do not superimpose.)

$$y = G[e] + S[d] \tag{1}$$

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Robot manipulators with tracking controllers are not infinitely stiff in response to external forces (also called disturbances). Even though the positioning controllers of robots are usually designed to follow the trajectory commands and reject dis-

turbances, the robot end-point will move somewhat in response to imposed forces on it. The motion of the robot end-point in response to imposed forces is caused by either structural compliance in the robot or by the compliance of the positioning controller. (In a simple example, if a Remote Centre Compliance (RCC) with a linear dynamic behaviour is installed at the end-point of the robot, then S is equal to the reciprocal of stiffness (impedance in the dynamic sense) of the RCC.) S is called the sensitivity function and it maps the external forces to the robot position. For a robot with a 'good' positioning controller, S is a mapping with small gain. (The gain of an operator is defined in Appendix A.) No assumption on the internal structures of G[e] and S[d]are made. Figure 1 shows the nature of the mapping in (1).

We assume that G[e] and S[d] are stable, non-linear operators in the L_p -space; in other words G[e] and S[d] are such that $G: L_p^n \mapsto L_p^n$, $S: L_p^n \mapsto L_p^n$ and also there exist constants α_1 , β_1 , α_2 , and β_2 such that $||G[e]||_p < \alpha_1 ||e||_p + \beta_1$ and $||S[d]||_p < \alpha_2 ||d||_p + \beta_2$. (The definition of stability in L_p -sense is given in Appendix A.)

Note that we modelled the dynamic behaviour of the robot based on an input-output functional relationship. This unified approach to modelling allows us to incorporate the dynamic behaviour of all the elements of the robot. Considering the robot as a rigid body, the dynamic behaviour of an open-loop robot can be derived by a set of non-linear differential equations via the lagrangian or eulerian approach. However, there may be enough components in the robot itself that rigid body dynamics is not sufficient for modelling. In fact, in many industrial hydraulic robots, the actuators and the servovalves dynamics dominate the total dynamic behaviour of the robots. We try to avoid using structured dynamic models such as first- or secondorder transfer functions as general representations of the dynamic behaviour of the components of the robot (e.g. servovalves in the hydraulic robots and the gear stiffness in the non-direct drive systems). We are proposing a dynamic model that can represent the complete dynamic behaviour of the robot in a very general form.

A similar modelling method can be given for analysis of the linearly treated robots. The transfer function matrices G and S in (2) are defined to describe the dynamic behaviour of a linearly treated robot manipulator with positioning controller:

$$y[j\omega] = G[j\omega]e[j\omega] + S[j\omega] d[j\omega]$$
(2)

(Throughout this paper, for the benefit of clarity, we develop the frequency domain theory for linearly treated robots in parallel with the non-linear analysis. The linear analysis is useful not only for analysis of robots with inherently linear dynamics, but also for robots with locally linearized dynamic behaviour. In the latter case, the analysis is correct only in the neighbourhood of the operating point.)

In (2), S is called the sensitivity transfer function matrix and it maps the external forces to the end-point position. $G[i\omega]$ is the closed-loop transfer function matrix that



Figure 1. Dynamics of the manipulator with the positioning controller (all the operators of the block diagrams of this paper are unspecified and may be frequency domain mappings or time domain input-output relationships).

maps the input trajectory vector, e, to the robot position, y. For a robot with a 'good' positioning controller, within the closed loop bandwidth $S[j\omega]$ is 'small' in the singular-value sense, while $G[j\omega]$ is approximately a unity matrix. (The maximum singular value of a matrix M, $\sigma_{max}[M]$ is defined as:

$$\sigma_{\max}[M] = \max \frac{|Mz|}{|z|}$$

where z is a non-zero vector and $|\cdot|$ denotes the euclidean norm.)

3. Dynamic behaviour of the environment

The environment can be very 'soft' or very 'hard'. We do not restrain ourselves to any geometry or to any structure. If one point on the environment is displaced as the vector x, then the required force to do such a task is f. Mapping E in (3) represents the environment dynamics:

$$f = E[x] \tag{3}$$

 x_0 in Fig. 2 is the initial location of the point of contact before deformation occurs and y is the robot end-point position $[x = y - x_0]$. E is assumed to be stable in L_p sense; $E: L_p^n \mapsto E: L_p^n$ and $||E(x)||_p < \alpha_3 ||x||_p + \beta_3$. Confining (3) to cover the linearly treated environment, (4) represents the dynamic behaviour of the environment with linear differential equations.

$$f[j\omega] = E[j\omega]x[j\omega]$$
(4)

 $E[j\omega]$ is a transfer function matrix that maps the amplitude of the displacement vector, x, to the amplitude of the contact force, f. The matrix E is an $n \times n$ transfer function matrix. E is a singular matrix when the robot interacts with the environment in only some directions. For example, in grinding a surface, the robot is constrained by the environment in the direction normal to the surface only. Readers can be convinced of the truth of (4) by analysing the relationship of the force and displacement of a



Figure 2. Environment and its dynamics.

spring as a simple model of the environment. E resembles the stiffness of a spring. Kazerooni (1986 a, b) represent $[Ms^2 + Cs + K]$ for E where M, C and K are symmetric square matrices and $s = j\omega$ (Lancaster *et al.* 1966). M is the positive definite inertia matrix while C and K are the positive semi-definite damping and the stiffness matrices, respectively.

4. Non-linear dynamic behaviour of the robot manipulator and environment

Suppose a manipulator with dynamic equation (1) is in contact with an environment given by (3); then f = -d. Figure 3 shows the dynamics of the robot manipulator and the environment when they are in contact with each other. Note that in some applications, the robot will have only uni-directional force on the environment. For example, in the grinding of a surface by a robot, the robot can only push the surface. If one considers positive f_i for 'pushing' and negative f_i for 'pulling', then in this class of manipulation, the robot manipulator and the environment are in contact with each other only along those directions where $f_i > 0$ for i = 1, ..., n. In some applications such as screwing a bolt, the interaction force can be positive and negative. This means the robot can have clockwise and counter-clockwise interaction torque. The non-linear discriminator block-diagram in Fig. 3 is drawn with a dashed line to illustrate the above concept.

Using (1)-(3), (5) and (6) represent the entire dynamic behaviour of the robot and environment as a whole:

$$y = G[e] + S[-f]$$
⁽⁵⁾

$$f = E[x] \quad \text{where } x = y - x_0 \tag{6}$$

If all the operators in Fig. 3 are considered linear transfer function matrices, (7) and (8) can be obtained to represent the end-point position and the contact force when $x_0 = 0$:

$$y = [I_n + SE]^{-1}Ge$$
(7)
$$f = E[I_n + SE]^{-1}Ge$$
(8)

$$f = E[I_n + SE]^{-1}Ge$$
(8)

To simplify the block diagram of Fig. 3, we introduce a mapping from e to f:

$$f = V[e] \tag{9}$$

V is assumed to be a stable operator in the L_p -sense; therefore: $V: L_p^n \mapsto L_p^n$ and also $||V[e]||_p < \alpha_4 ||e||_p + \beta_4$. Note that one can still define V without assuming the superposition of effects of e and d in (5) (or (1)). If all the operators in Fig. 3 are transfer function matrices, then $V = E[I_n + SE]^{-1}G$.



Figure 3. Interaction of the robot manipulator with the environment.

5. Architecture of the closed-loop system

We propose the architecture of Fig. 4 to develop compliancy for the robot. The compensator, H, is considered to operate on the contact force, f. The compensator output signal is being subtracted from the input command vector, r, resulting in the input trajectory vector, e, for the robot manipulator.

There are two feedback loops in the system; the inner loop (which is the natural feedback loop), is the same as the one shown in Fig. 3. This loop shows how the contact force affects the robot in a natural way when the robot is in contact with the environment. The outer feedback loop is the controlled feedback loop. If the robot and the environment are not in contact, then the dynamic behaviour of the system reduces to the one represented by (1), which is a plain positioning system. When the robot and the environment are in contact, then the value of the contact force and the end-point position of robot are given by f and y where the following equations are true:

$$y = G[e] + S[-f]$$
 (10)

$$f = E[x] \quad \text{where } x = y - x_0 \tag{11}$$

$$e = r - H[f] \tag{12}$$

If the operators in (10), (11), and (12) are considered transfer function matrices, (13) and (14) can be obtained to represent the interaction force and the robot endpoint trajectory for linearly treated systems when $x_0 = 0$:

$$f = E[I_n + SE + GHE]^{-1}Gr$$
(13)

$$y = [I_n + SE + GHE]^{-1}Gr$$
(14)

The objective is to choose a class of compensators, H, to control the contact force with the input command r. By knowing S, G, E, and choosing H, one can shape the contact force. The value of H is the choice of the designer and, depending on the task, it can have various values in different directions. A large value for H develops a compliant robot while a small H generates a stiff robot. Note that S and GH add in (14) to develop the total compliancy in the system. GH represents the electronic compliancy in the robot structural compliancy) in the system. Equation (14) can be rewritten as $y = E^{-1}[E^{-1} + S + GH]^{-1}Gr$. Note that the environment admittance (1/impedance in the linear domain), E^{-1} , the robot sensitivity (1/stiffness in the linear domain), S, and



Figure 4. Closed-loop system.

the electronic compliancy, GH add together to form the total sensitivity of the system. If H = 0, then only the admittance of the environment and the robot add together to form the compliancy for the system. By closing the loop via H, one can not only add to the total sensitivity but also shape the sensitivity of the system. One cannot choose arbitrarily large values for H; the stability of the closed-loop system of Fig. 4 must be guaranteed. The trade-off between the closed-loop stability and the size of H is investigated in § 6.

When the robot is not in contact with the environment (i.e. the outer feedback loop in Fig. 4 does not exist), the actual position of the robot end-point is governed by (1). When the robot is in contact with the environment, then the contact force follows r according to (10), (11) and (12). The input command vector, r, is used differently for the two categories of manoeuvrings; as an input trajectory command in unconstrained space (see (1)) and as a command to control force in constrained space. We do not command any set-point for force as we do in admittance control (Raibert and Craig 1981, Whitney 1977). This method is called *impedance control* (Hogan 1985, Kazerooni 1986 a, b) because it accepts a position vector as input and reflects a force vector as output. There is no hardware or software switch in the control system when the robot travels between unconstrained space and constrained space. The feedback loop on the contact force closes naturally when the robot encounters the environment.

6. Stability analysis

The objective of this section is to arrive at a sufficient condition for stability of the system shown in Fig. 4. This sufficient condition leads to the introduction of a class of compensators, H, that can be used to develop compliancy for the family of robot manipulators with dynamic behaviour represented by (1). Using operator V defined by (9), the block diagram of Fig. 5 is constructed as a simplified version of the block diagram of Fig. 4. First we use the Small Gain Theorem to derive the general stability condition. Then, with the help of a corollary, we show the stability condition when His chosen as a linear operator (transfer function matrix). Inequality (24) shows the bound on the size of H in the singular-value sense when H is a transfer function matrix while V is still a non-linear operator. Finally, if all the operators in Fig. 4 are transfer function matrices, then the stability bound is shown by (25). Section 7 is devoted to stability analysis of the linearly treated systems, when the environment is infinitely rigid in comparison with the robot stiffness. The stability analysis and the role of robot sensitivity and environment dynamics on size H are best shown by linear theory in (27)-(31). In particular, we confine our analysis to linear one-degree-of-freedom robot in (32) and (33) for better understanding the nature of the stability analysis.

The following proposition (using the Small Gain Theorem) states the stability condition of the closed-loop system shown in Fig. 5.



Figure 5. Manipulator and the environment with force feedback compensator (simplified version of Figure 4).

If conditions (i), (ii) and (iii) hold:

(i) V is an L_p -stable operator, that is

$$(a) \quad V[e]: L_p^n \mapsto L_p^n \tag{15}$$

(b)
$$\|V[e]\|_{p} < \alpha_{4} \|e\|_{p} + \beta_{4}$$
 (16)

(ii) H is chosen such that mapping H[f] is L_p -stable, that is

(a)
$$H[f]: L_p^n \mapsto L_p^n$$

(b) $\|H[f]\|_p < \alpha_5 \|f\|_p + \beta_5$
and
(iii) $\alpha_4 \alpha_5 < 1$
(19)

then the closed-loop system (Fig. 5) is L_p -stable. The proof is given in Appendix A. Substituting for $||f||_p$ from (16) into (18) results in (20) (note that f = V[e]).

$$\|HV[e]\|_{p} < \alpha_{4}\alpha_{5} \|e\|_{p} + \alpha_{5}\beta_{4} + \beta_{5}$$

$$\tag{20}$$

 $\alpha_4 \alpha_5$ in (20) represents the gain of the loop mapping, HV[e]. The third stability condition requires that H be chosen such that the loop mapping, HV[e], is linearly bounded with less than a unity slope. The following corollary develops a stability bound if H is selected as a linear transfer function matrix.

Corollary

The key parameter in the proposition is the size of $\alpha_4 \alpha_5$. According to the proposition, to guarantee the stability of the system, *H* must be chosen such that norm of HV[e] is linearly bounded with a slope that is smaller than unity. If *H* is chosen as a linear operator (the impulse response) while all the other operators are still non-linear, then:

$$\|HV[e]\|_{p} < \gamma \|V[e]\|_{p}$$

where

$$\gamma = \sigma_{\max}[N] \tag{22}$$

 σ_{\max} indicates the maximum singular value, and N is a matrix whose *ij*th entry is $||H_{ij}||_1$. In other words, each member of N is the L_1 norm of each corresponding member of H. Considering (16), (21) can be rewritten as:

$$|HV[e]||_{p} < \gamma ||V[e]||_{p} < \gamma \alpha_{4} ||e||_{p} + \gamma \beta_{4}$$

$$\tag{23}$$

Comparing (23) with (20), to guarantee the closed-loop stability, $\gamma \alpha_4$ must be smaller than unity, or, equivalently:

$$\gamma < \frac{1}{\alpha_4}$$

To guarantee the stability of the closed loop system, H must be chosen such that its 'size' is smaller than the reciprocal of the 'gain' of the forward loop mapping in Fig. 5. Note that γ represents a 'size' of H in the singular-value sense.

When all the operators of Fig. 5 are linear transfer function matrices one can use Multivariable Nyquist Criteria to arrive at the sufficient condition for stability of the closed-loop system. This sufficient condition leads to the introduction of a class of transfer function matrices, H, that stabilize the family of linearly treated robot manipulators and environment using dynamic equations (2) and (4). The detailed derivation for the stability condition is given in Appendix C. Appendix D shows that the stability condition given by Nyquist Criteria is a subset of the condition given by the Small Gain Theorem. According to the results of Appendix C, the sufficient condition for stability is given by (25).

$$\sigma_{\max}[GHE] \leq \sigma_{\min}[SE + I_n] \quad \text{for all } \omega \in [0, \infty]$$
(25)

or a more conservative condition,

$$\sigma_{\max}[H] < \frac{1}{\sigma_{\max}[E[I_n + SE]^{-1}G]} \quad \text{for all } \omega \in [0, \infty]$$

Similar to the non-linear case, H must be chosen such that its 'size' is smaller than the reciprocal of the 'size' of the forward loop mapping in Fig. 6 to guarantee the stability of the closed-loop system. Note that in (26) σ_{max} represents a 'size' of H in the singular-value sense.

Inequality (26) reveals some facts about the size of H. The smaller the sensitivity of the robot manipulator is, the smaller H must be chosen. Also from (26), the more rigid the environment is, the smaller H must be chosen. In the 'ideal case', no H can be found to allow a perfect positioning system [S = 0] to interact with an infinitely rigid environment $[E = \infty]$. In other words, for stability of the system shown in Fig. 4, there must be some compliancy either in robot or in the environment. RCC, structural dynamics and the tracking controller stiffness form the compliancy of the robot. Section 7 gives more information about the effects of E on the stability region.



Figure 6. Simplified form of Fig. 5 when all the operators are linear transfer function matrices: $V = E(I_n + SE)^{-1}G.$

7. Stability for very rigid environment

In most manufacturing tasks, the end-point of the robot manipulator is in contact with a very stiff environment. Robotic deburring and grinding are examples of practical tasks in which the robot is in contact with hard environment (Kazerooni 1986 c, 1986 a, 1988). According to the results in Appendix B, when the environment is very stiff, (E is very 'large' in the singular-value sense), the limiting value for the contact force and the end-point position are given by (27) and (28) respectively:

$$f_{\infty} = [S + GH]^{-1}Gr \tag{27}$$

$$y_{\infty} = 0 \tag{28}$$

Since $G \approx I_n$ for all $\omega \in [0, \omega_0]$, (the end-point position is 'approximately' equal to the

input trajectory vector, e), the value of the contact force, f, within the bandwidth of the system $[0, \omega_0]$ can be approximated by (29):

$$f_{\infty} \approx [S + H]^{-1}r$$
 for all $\omega \in [0, \omega_0]$ (29)

By knowing S and choosing H, one can shape the contact force. The value of [S + H] within $[0, \omega_0]$ is the designer's choice and, depending on the task, it can have various values in different directions (Kazerooni 1986 a, 1986 b). A large value for [S + H] within $[0, \omega_0]$ develops a compliant system while a small [S + H] generates a stiff system. If H is chosen such that [S + H] is 'large' in the singular-value sense at high frequencies, then the contact force in response to high-frequency components of r will be small. If H is chosen to guarantee the compliance in the system according to (29), then it must also satisfy the stability condition. It can be shown that the stability criteria for interaction with a very rigid environment is given by (30):

$$\sigma_{\max}[H] \leq \frac{1}{\sigma_{\max}[S^{-1}G]} \quad \text{for all } \omega \in [0, \infty]$$
 (50)

It is clear that if the environment is very rigid, then one must choose a very small H to satisfy the stability of the system when S is 'small'. (A good positioning system has 'small' S.) Since $G \approx I_n$ for all $\omega \in [0, \omega_0]$, the bound for H, for a rigid environment and a 'small' stiffness, is given by (31):

$$\sigma_{\max}[H] \leq \sigma_{\min}[S] \quad \text{for all } \omega \in [0, \omega_0] \tag{31}$$

If S is zero, then no H can be obtained to stabilize the system. To stabilize the system of the very rigid environment and the robot, there must be a minimum compliancy in the robot. Direct drive manipulators, because of the elimination of the transmission systems, often have large S. This allows for a wider stability range in constrained manipulation. In the case of the one-degree-of-freedom system the condition for stability is given by (32):

$$|HG| < \left| \left[S + \frac{1}{E} \right] \right|$$
 for all $\omega \in [0, \infty]$ (32)

where $|\cdot|$ denotes the magnitude of a transfer function. Since in many cases $G \approx 1$ for all $0 < \omega < \omega_0$, then S must be chosen such that:

$$|H| < \left| \left[S + \frac{1}{E} \right] \right| \quad \text{for all } \omega \in [0, \omega_0] \tag{33}$$

Inequality (33) clearly shows that the more rigid the environment is, the smaller H must be chosen to guarantee the stability of the closed-loop system. In the case of a rigid environment ('large' E) and a 'good' positioning system, H must be chosen as a very small gain.

We conclude that for stability of the environment and the robot taken as a whole, there must be some initial compliancy either in the robot or in the environment. The initial compliancy in the robot can be obtained by a non-zero sensitivity function or a passive compliant element such as an RCC. Practitioners always observed that the system of a robot and a stiff environment can always be stabilized when a compliant element (e.g. piece of rubber or RCC) is installed between the robot and environment. One can also stabilize the system of robot and environment by increasing the robot sensitivity function. In many commercial manipulators the sensitivity of the robot

manipulators can be increased by decreasing the gain of each actuator positioning loop. This also results in a narrower bandwidth (slow response in the unconstrained manoeuvring) for the robot positioning system.

8. Summary and conclusion

A new controller architecture for compliance control has been investigated using unstructured models for dynamic behaviour of robot manipulators and environment. This unified approach of modelling robot and environment dynamics is expressed in terms of sensitivity functions. The control approach allows not only for tracking the input-command vector, but also for compliancy in the constrained manoeuvrings. A bound for the global stability of the manipulator and environment has been derived. For stability of the environment and the robot taken as a whole, there must be some initial compliancy either in the robot or in the environment. The initial compliancy in the robot can be obtained by a non-zero sensitivity function for the positioning controller or a passive compliant element such as an RCC.

Appendix A

Definitions 1 to 7 will be used in the stability proof of the closed-loop system (Vidyasagar and Desoer 1975, Vidyasagar 1978).

Definition 1

For all $p \in [1, \infty]$, we label as L_p^n the set consisting of all functions $f = [f_1 \ f_2 \ \dots \ f_n]^T : [0, \infty] \mapsto \mathbb{R}^n$ such that:

$$\int_0^\infty |f_i|^p dt < \infty \quad \text{for } i = 1, 2, ..., n$$

Definition 2

For all $T \in [0, \infty]$, the function f_T defined by:

$$f_T = \begin{cases} f & 0 \le t \le T \\ 0 & T < t \end{cases}$$

is called the truncation of f to the interval [0, T].

Definition 3

The set of all functions $f \models [f_1 \ f_2 \ \dots \ f_n]^T : [0, \infty] \mapsto \mathbb{R}^n$ such that $f^T \in L_p^n$ for all finite T is denoted by L_{pe}^n . f by itself may or may not belong to L_p^n .

Definition 4

The norm on L_p^n is defined by:

$$\|f\|_{p} = \left[\sum_{i=1}^{n} \|f_{i}\|_{p}^{2}\right]^{1/2}$$

where $||f_i||_p$ is defined as:

$$\|f_i\|_p = \left[\int_0^\infty \omega_i |f_i|^p dt\right]^{1/p}$$

where ω_i is the weighting factor. ω_i is particularly useful for scaling forces and torques of different units.

Definition 5

Let $v[\cdot]: L_{pe}^n \mapsto L_{pe}^n$. We say that the operator $V[\cdot]$ is L_p -stable, if:

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(a) v[\cdot]: L_p^n \mapsto L_p^n
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(b) there exist finite real constants α_4 and β_4 such that:

$$\|V[e]\|_p < \alpha_4 \|e\|_p + \beta_4 \quad \forall e \in L_p^n$$

According to this definition we first assume that the operator maps L_{pe}^{n} to L_{pe}^{n} . It is clear that if one does not show that $v[\cdot]: L_{pe}^{n} \mapsto L_{pe}^{n}$, the satisfaction of condition (a) is impossible since L_{pe}^{n} contains L_{p}^{n} . Once mapping, $v[\cdot]$, from L_{pe}^{n} to L_{pe}^{n} is established, then we say that the operator $v[\cdot]$ is L_{p} -stable if, whenever the input belongs to L_{p}^{n} . Moreover, the norm of the output is no larger than α_{4} times the norm of the input plus the offset constant β_{4} .

Definition 6

The smallest α_4 such that there exist a β_4 so that inequality b of Definition 5 is satisfied is called the gain of the operator $v[\cdot]$.

Definition 7

Let $V[\cdot]: L_{pe}^{n} \mapsto L_{pe}^{n}$. The operator $V[\cdot]$ is said to be causal if:

$$V[e]_{\mathsf{T}} = V[e_{\mathsf{T}}] \quad \forall \, T < \infty \quad \text{and} \quad \forall \, e \in L_{pe}^{n}$$

Proof of the non-linear stability proposition

Define the closed-loop mapping $A: r \mapsto e$ (Fig. 5):

$$e = r - HV[e]$$

For each finite T, (A 2) is true.

$$||e_T||_p < ||r_T||_p + ||HV[e]_T||_p$$
 for all $t \in [0, T]$

Since HV[e] is L_p -stable. Therefore, (A 3) is true.

$$||e_T||_p < ||r_T||_p + \alpha_5 \alpha_4 ||e_T||_p + \alpha_5 \beta_4 + \beta_5$$
 for all $t \in [0, T]$

Since $\alpha_5 \alpha_4$ is less than unity:

$$\|e_{T}\|_{p} < \frac{\|r_{T}\|_{p}}{1 - \alpha_{5}\alpha_{4}} + \frac{\alpha_{5}\beta_{4} + \beta_{5}}{1 - \alpha_{5}\alpha_{4}} \quad \text{for all } t \in [0, T]$$
 (A 4)

Inequality (A 4) shows that $e[\cdot]$ is bounded over [0, T]. Because this reasoning is valid for every finite *T*, it follows that $e[\cdot] \in L_{pe}^{n}$, i.e. that $A: L_{pe}^{n} \mapsto L_{pe}^{n}$. Next we show that the mapping *A* is L_{p} -stable in the sense of Definition 5. Since $r \in L_{p}^{n}$, therefore $||r||_{p} < \infty$ for all $t \in [0, \infty]$, therefore (A 5) is true:

$$\|e\|_{p} < \infty \quad \text{for all } t \in [0, \infty] \tag{A 5}$$

Inequality (A 5) implies e belongs to L_p -space whenever r belong to L_p -space. With the

same reasoning from (A 1)-(A 5), it can be shown that (A 6) is true:

$$\|e\|_{p} < \frac{\|r\|_{p}}{1 + \alpha_{5}\alpha_{4}} + \frac{\alpha_{5}\beta_{4} + \beta_{5}}{1 - \alpha_{5}\alpha_{4}} \quad \text{for all } t \in [0, T]$$
 (A 6)

Inequality (A 6) shows the linear boundedness of e (Condition (b) of Definition 5). Inequality (A 6) and (A 5) taken together, guarantee that the closed-loop mapping A is L_p -stable.

Appendix B

A very rigid environment generates a very large force for a small displacement. We choose the minimum singular value of E to represent the size of E. The following proposition states the limiting value of the force when the robot manipulator is in contact with a very rigid environment.

If $\sigma_{\min}[E] > M_0$, where M_0 is an arbitrarily large number, then the value of the force given by (21) will approach the expression given by (B 1):

$$f_{\infty} = [S + GH]^{-1} Gr \tag{B1}$$

Proof

We will prove that $|f_{\infty} - f|$ approaches a small number as M_0 approaches a large number:

$$f_{\infty} - f = [S + GH]^{-1} [I_n - [S + GH]E[I_n + SE + GHE]^{-1}]Gr$$

Factoring $[I_n + SE + GHE]^{+1}$ to the right-hand side:

$$f_{\infty} - f = [S + GH]^{-1} [I_{n} + SE + GHE]^{-1} Gr$$

$$|f_{\infty} - f| < \sigma_{\max} [S + GH]^{-1} \times \sigma_{\max} [I_{n} + SE + GHE]^{-1} \times \sigma_{\max} [G]|r|$$

$$|f_{\infty} - f| < \frac{\sigma_{\max} [G]|r|}{\sigma_{\min} [S + GH] \times [\sigma_{\min} [SE + GHE] - 1]}$$

$$|f_{\infty} - f| < \frac{\sigma_{\max} [G]|r|}{\sigma_{\min} [S + GH] \times [\sigma_{\min} [S + GH] \times \sigma_{\min} [E] - 1]}$$
(B 5)

 $\sigma_{\max}[G]$ and $\sigma_{\min}[S + GH]$ are bounded values. If $\sigma_{\min}[E] > M_0$, then it is clear that the left-hand side of (B 6) can be an arbitrarily small number by choosing M_0 to be a large number. (The proof for $y_{\infty} \approx 0$ is similar to the above.)

Appendix C

The objective is to find a sufficient condition for stability of the closed-loop system in Fig. 4 by Nyquist criteria. The block diagram in Fig. 4 can be reduced to the block diagram in Fig. 7 when all the operators are linear transfer function matrices and $x_0 = 0$.

There are two elements in the feedback loop; *GHE* and *SE*. *SE* shows the natural force feedback while *GHE* represents the controlled force feedback in the system. If H = 0, then the system in Fig. 7 reduces to the system in Fig. 3 (a stable positioning robot manipulator which is in contact with the environment *E*). The objective is to use Nyquist criteria (Lehtomaki 1981, Kazerooni and Houpt 1986) to arrive at the



Figure 7. Simplified block-diagram of the system in Fig. 5.

sufficient condition for stability of the system when H = 0. The following conditions are regarded.

- (i) The closed-loop system in Fig. 7 is stable if H = 0. This condition simply states the stability of the robot manipulator and environment when they are in contact. (Figure 3 shows this configuration.)
- (ii) H is chosen as a stable linear transfer function matrix. Therefore the augmented loop transfer function [GHE + SE] has the same number of unstable poles that SE has. Note that in many cases SE is a stable system.
- (iii) Number of poles on $j\omega$ axis for both loop SE and [GHE + SE] are equal.

Considering that the system in Fig. 7 is stable when H = 0, we plan to find how robust the system is when GHE is added to the feedback loop. If the loop transfer function SE (without compensator, H) develops a stable closed-loop system, then we are looking for a condition on H such that the augmented loop transfer function [GHE + SE] guarantees the stability of the closed-loop system. According to the Nyquist criteria, the system in Fig. 7 remains stable if the anticlockwise encirclement of the det $[SE + GHE + I_n]$ around the centre of the s-plane is equal to the number of unstable poles of the loop transfer function [GHE + SE]. According to Conditions (ii) and (iii) the loop transfer functions SE and [GHE + SE] both have the same number of unstable poles. The closed-loop system when H=0 is stable according to Condition (i); the encirclements of det $[SE + I_n]$ is equal to unstable poles of SE. When GHE is added to the system, for stability of the closed-loop system, the number of the encirclements of det $[SE + GHE + I_n]$ must be equal to the number of unstable poles of the [GHE + SE]. Since the number of unstable poles of [SE + GHE] and SE are the same, therefore for stability of the system det $[SE + GHE + I_n]$ must have the same number of encirclements that det $[SE + I_n]$ has. A sufficient condition to guarantee the equality of the number of encirclements of det $[SE + GHE + I_n]$ and det $[SE + I_n]$ is that the det $[SE + GHE + I_n]$ does not pass through the origin of the s-plane for all possible non-zero but finite values of H, or

det
$$[SE + GHE + I_n] \neq 0$$
 for all $\omega \in [0, \infty]$ (C 1)

If (C 1) does not hold then there must be a non-zero vector z such that:

 $[SE + GHE + I_n]z = 0 \tag{C 2}$

or

$$GHEz = -[SE + I_n]z \tag{C 3}$$

A sufficient condition to guarantee that (C 3) will not occur is given by (C 4).

 $\sigma_{\max}[GHE] \leq \sigma_{\min}[SE + I_n] \quad \text{for all } \omega \in [0, \infty]$

or a more conservative condition:

$$\sigma_{\max}[H] \leq \frac{1}{\sigma_{\max}[E(SE+I_n)^{-1}G]} \quad \text{for all } \omega \in [0, \infty] \qquad (\bigcirc \Im)$$

Note that $E[SE + I_n]^{-1}G$ is the transfer function matrix that maps *e* to the contact force, *f*. Figure 6 shows the closed-loop system. According to the result of the proposition, *H* must be chosen such that the size of *H* is smaller than the reciprocal of the size of the forward loop transfer function, $E[SE + I_n]^{-1}G$.

Appendix D

The following inequalities are true when p = 2 and H and V are linear operators:

$$||H[V[e]]||_{p} < v ||V[e]||_{p}$$
 (D 1)

$$\|V[e]\|_{p} < \mu \|e\|_{p}$$
 (D 2)

where

 $\mu = \sigma_{\max}[Q]$, and Q is the matrix whose *ij*th entry is given by $[Q]_{ij} = \sup_{\omega} |[Q]_{ij}|$ $\nu = \sigma_{\max}[R]$, and R is the matrix whose *ij*th entry is given by $[R]_{ij} = \sup_{\omega} |[R]_{ij}|$

Substituting inequality (D 2) in (D 1):

$$||HV[e]||_{p} < \mu v ||e||_{p}$$
 (D 3)

According to the stability condition, to guarantee the closed-loop stability $\mu\nu < 1$ or:

 $v < \frac{1}{\mu}$ (D 4)

Note that the following are true:

0

$$w_{\max}[V] < \mu \quad \text{for all } \omega \in [0, \infty] \tag{D 5}$$

$$\sigma_{\max}[H] < v \quad \text{for all } \omega \in [0, \infty] \tag{D 6}$$

Substituting (D 5) and (D 6) into (D 4) which guarantees the stability of the system, the following inequality is obtained:

$$\sigma_{\max}[H] < \frac{1}{\sigma_{\max}[V]} \quad \text{for all } \omega \in [0, \infty]$$

$$\sigma_{\max}[H] < \frac{1}{\sigma_{\max}[E[I_n + SE]^{-1}G]} \quad \text{for all } \omega \in [0, \infty]$$

Inequality (D 8) is identical to (26). This shows that the linear condition for stability given by the multivariable Nyquist criterion is a subset of the general condition given by the small gain theorem.

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